

Unique Factorization and Controllability of Tail-Biting Trellis Realizations via Controller Granule Decompositions

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Abstract—The Conti-Boston factorization theorem (CBFT) for linear tail-biting trellis realizations is extended to group realizations with a new and simpler proof, based on a controller granule decomposition of the behavior and known controllability results for group realizations. Further controllability results are given; e.g., a trellis realization is controllable if and only if its top (controllability) granule is trivial.

I. INTRODUCTION

Tail-biting trellis realizations are the simplest class of realizations of codes on cyclic graphs. Decoding is generally simpler than for conventional trellis realizations [1].

Koetter and Vardy [8], [9] developed the foundations of the theory of linear tail-biting trellis realizations. Their key result was a factorization theorem (KVFT), which shows that every reduced realization has a factorization into elementary trellises.

Recently, Conti and Boston [2] have proved a stronger unique factorization theorem (CBFT): the behavior (“label code”) of a reduced linear tail-biting trellis realization factors uniquely into quotient spaces of “span subcodes.” This work was the main stimulus for the work reported here.

Our main result is a generalization of the CBFT to group realizations, with a new proof that we feel is even simpler and more insightful. [2, Remark III.3] notes that such a generalization is not straightforward.

In Section II, we introduce a granule decomposition along the lines of the controller granule decomposition of minimal conventional trellis realizations of Forney and Trott [5], [6], and the span subcode decomposition of [2].

In Section III, using results of [3] on the controllability of group realizations, we show that this granule decomposition yields a unique factorization of a group trellis behavior \mathfrak{B} . We develop other controllability properties not considered in [2]; e.g., the trellis diagram of an uncontrollable group trellis realization is disconnected [4]. We show that the controller canonical realization based on this factorization is one-to-one, minimal, and group-theoretic, but possibly nonhomomorphic.

Our development uses only elementary group theory, principally the *fundamental theorem of homomorphisms* (FTH) and the *correspondence theorem* (CT). For a brief introduction to the necessary group theory and our notation, see [3].

A. Preliminaries

A (tail-biting) trellis realization \mathcal{R} of length n is defined by a set of n symbol alphabets $\{\mathcal{A}_j, j \in \mathbb{Z}_n\}$, a set of n state alphabets $\{\mathcal{S}_j, j \in \mathbb{Z}_n\}$, and a set of n constraint codes $\{\mathcal{C}_j \subseteq \mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}, j \in \mathbb{Z}_n\}$, where index arithmetic is in \mathbb{Z}_n ; e.g., $\mathcal{C}_{n-1} \subseteq \mathcal{S}_{n-1} \times \mathcal{A}_{n-1} \times \mathcal{S}_0$.

The configuration universe $\mathcal{U} = \prod_{j \in \mathbb{Z}_n} \mathcal{C}_j$ is thus a subset of $\mathcal{S} \times \mathcal{A} \times \mathcal{S}$, where $\mathcal{A} = \prod_{j \in \mathbb{Z}_n} \mathcal{A}_j$ and $\mathcal{S} = \prod_{j \in \mathbb{Z}_n} \mathcal{S}_j$.

In a linear trellis realization, each symbol or state alphabet is a finite-dimensional vector space over some field \mathbb{F} , and each \mathcal{C}_j is a subspace of $\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}$, so \mathcal{U} is a subspace of $\mathcal{S} \times \mathcal{A} \times \mathcal{S}$. (In [9] and [2], it is assumed that $\mathcal{A}_j = \mathbb{F}$ always.) In a group trellis realization, each symbol or state alphabet is a finite abelian group, and each \mathcal{C}_j is a subgroup of $\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}$, so \mathcal{U} is a subgroup of $\mathcal{S} \times \mathcal{A} \times \mathcal{S}$.

The extended behavior \mathfrak{B} of \mathcal{R} is the set of configurations $(s, a, s') \in \mathcal{U}$ such that $s = s'$; i.e., such that the constraints of \mathcal{U} and the equality constraints $s = s'$ are both satisfied [3]. Its behavior \mathfrak{B} is the projection of \mathfrak{B} onto $\mathcal{A} \times \mathcal{S}$, which is an isomorphism. The code \mathcal{C} realized by \mathcal{R} is the projection of \mathfrak{B} or \mathfrak{B} onto \mathcal{A} .

The (normal) graph of \mathcal{R} [3] is the single-cycle graph with n vertices corresponding to the constraint codes \mathcal{C}_j , n edges corresponding to the state variables \mathcal{S}_j , where edge \mathcal{S}_j is incident on vertices \mathcal{C}_{j-1} and \mathcal{C}_j , and n half-edges corresponding to the symbol variables \mathcal{A}_j , where half-edge \mathcal{A}_j is incident only on vertex \mathcal{C}_j .

II. GRANULE DECOMPOSITION

A. Partial ordering of fragments

A proper fragment of a trellis realization \mathcal{R} corresponds to a circular interval $[j, k)$, $j \in \mathbb{Z}_n$, $k \in \mathbb{Z}_n$, and will be denoted by $\mathcal{F}^{[j,k)}$. $\mathcal{F}^{[j,k)}$ includes the constraint codes $\{\mathcal{C}_{j'}, j' \in [j, k)\}$ and the internal state variables $\{\mathcal{S}_{j'}, j' \in (j, k)\}$, and has boundary $\{\mathcal{S}_j, \mathcal{S}_k\}$. Accordingly, we define its vertex set as $V(\mathcal{F}^{[j,k)}) = [j, k)$, and its edge set as $E(\mathcal{F}^{[j,k)}) = (j, k)$. The (normal) graph of every proper fragment is cycle-free.

We define the level of $\mathcal{F}^{[j,k)}$ as the number $\ell = |E(\mathcal{F}^{[j,k)})|$ of its internal state variables; i.e., $\ell = k - j - 1 \bmod n$.

Thus $|V(\mathcal{F}^{[j,k]})| = \ell + 1$. We may denote a level- ℓ fragment $\mathcal{F}^{[j,j+\ell+1]}$ by $\mathcal{F}^{[j,j+\ell]}$. A level- $(n-1)$ fragment $\mathcal{F}^{[j,j]}$ is obtained from \mathcal{R} by cutting the edge \mathcal{S}_j into two half-edges; it contains all n constraint codes and $n-1$ internal state variables. A level-0 fragment $\mathcal{F}^{[j,j+1]} = \mathcal{F}^{[j,j]}$ contains one constraint code \mathcal{C}_j and no internal state variables.

We also regard the entire realization \mathcal{R} as a fragment, whose level is n . \mathcal{R} contains $\ell = |E(\mathcal{R})| = n$ internal state variables, and $\ell = |V(\mathcal{R})| = n$ (not $\ell + 1$) constraint codes.

As observed in [2], the set $\mathfrak{F}(\mathcal{R})$ of fragments of a tail-biting trellis realization \mathcal{R} is partially ordered by set inclusion. The maximum fragment \mathcal{R} includes all proper fragments $\mathcal{F}^{[j,k]}$. The partial ordering of proper fragments corresponds to the partial ordering of the circular intervals $[j, k]$ by set inclusion; i.e., $\mathcal{F}^{[j',k']} \leq \mathcal{F}^{[j,k]}$ iff $[j', k'] \subseteq [j, k]$. The minimal fragments are the level-0 fragments $\mathcal{F}^{[j,j+1]}$.

The partial ordering of $\mathfrak{F}(\mathcal{R})$ may be illustrated by a *Hasse diagram*, as follows. A fragment $\mathcal{F}' \in \mathfrak{F}(\mathcal{R})$ is said to be *covered* by another fragment $\mathcal{F} \in \mathfrak{F}(\mathcal{R})$ if $\mathcal{F}' < \mathcal{F}$ and there is no fragment $\mathcal{F}'' \in \mathfrak{F}(\mathcal{R})$ such that $\mathcal{F}' < \mathcal{F}'' < \mathcal{F}$ [10]. In our setting, \mathcal{F}' is covered by \mathcal{F} if $\mathcal{F}' < \mathcal{F}$ and the level of \mathcal{F}' is one less than the level of \mathcal{F} . The set $\mathfrak{F}(\mathcal{R})$ is thus said to be *graded* by level (number of internal state variables).

The Hasse diagram of $\mathfrak{F}(\mathcal{R})$ is illustrated in Figure 1 for a tail-biting trellis realization \mathcal{R} of length $n = 4$.

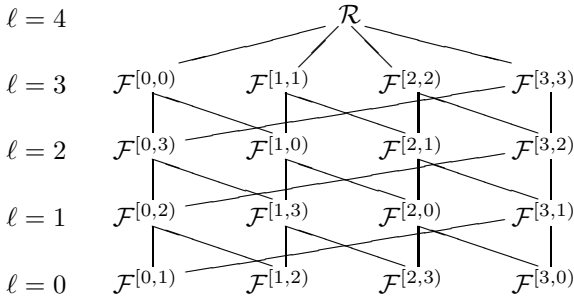


Fig. 1. Hasse diagram of $\mathfrak{F}(\mathcal{R})$ when $n = 4$.

As numerous authors have observed (e.g., [9], [2]), a conventional trellis realization may be viewed as a special case of a tail-biting trellis realization in which \mathcal{S}_0 is trivial. Correspondingly, the Hasse diagram of a conventional trellis realization is a subdiagram of the Hasse diagram for a tail-biting trellis realization \mathcal{R} of the same length, comprising the fragments $\{\mathcal{F} \in \mathfrak{F}(\mathcal{R}) \mid \mathcal{F} \leq \mathcal{F}^{[0,0]}\}$. By cyclic rotation of the index set \mathbb{Z}_n , any level- $(n-1)$ fragment $\mathcal{F}^{[j,j]}$ may be regarded as a conventional trellis realization.

B. Subbehaviors

For every proper fragment $\mathcal{F} = \mathcal{F}^{[j,k]} \in \mathfrak{F}(\mathcal{R})$, we define the *subbehavior* $\mathfrak{B}_{\mathcal{F}} = \mathfrak{B}^{[j,k]}$ as the set of $(\mathbf{a}, \mathbf{s}) \in \mathfrak{B}$ that are all-zero on or outside the boundary of \mathcal{F} . For example, $\mathfrak{B}^{[0,0]}$ is the behavior of a conventional trellis realization of length n . We also define $\mathfrak{B}_{\mathcal{R}} = \mathfrak{B}$.

Evidently if $\mathcal{F}' \leq \mathcal{F}$, then $\mathfrak{B}_{\mathcal{F}'} \subseteq \mathfrak{B}_{\mathcal{F}}$. Thus the set $\{\mathfrak{B}_{\mathcal{F}}, \mathcal{F} \in \mathfrak{F}(\mathcal{R})\}$ has the same partial ordering as $\mathfrak{F}(\mathcal{R})$.

For a level-0 fragment $\mathcal{F}^{[j,j]}$, we have

$$\mathfrak{B}^{[j,j]} = \{(\mathbf{a}, \mathbf{0}) \mid a_j \in (\mathcal{C}_j)_{:\mathcal{A}_j}, a_{j'} = 0 \text{ if } j' \neq j\},$$

where $(\mathcal{C}_j)_{:\mathcal{A}_j} = \{a_j \in \mathcal{A}_j \mid (0, a_j, 0) \in \mathcal{C}_j\}$ is the *cross-section* of \mathcal{C}_j on \mathcal{A}_j . As in [3], $(\mathcal{C}_j)_{:\mathcal{A}_j}$ will be denoted by $\underline{\mathcal{A}}_j$, and called the *nondynamical symbol alphabet* of \mathcal{C}_j .

C. Granules

For non-level-0 fragments, we define $\mathfrak{B}_{<\mathcal{F}}$ as the behavior generated by all $\mathfrak{B}_{\mathcal{F}'}$ such that $\mathcal{F}' < \mathcal{F}$, as in [2]. In other words, $\mathfrak{B}_{<\mathcal{F}} = \sum_{\mathcal{F}' < \mathcal{F}} \mathfrak{B}_{\mathcal{F}'}$. Evidently $\mathfrak{B}_{<\mathcal{F}} \subseteq \mathfrak{B}_{\mathcal{F}}$.

We define the *controller granule* $\Gamma_{\mathcal{F}}$ as the quotient $\mathfrak{B}_{\mathcal{F}}/\mathfrak{B}_{<\mathcal{F}}$. In the linear case, $\mathfrak{B}_{\mathcal{F}}$ and $\mathfrak{B}_{<\mathcal{F}}$ are vector spaces, and their quotient $\Gamma_{\mathcal{F}}$ is a vector space of dimension $\dim \Gamma_{\mathcal{F}} = \dim \mathfrak{B}_{\mathcal{F}} - \dim \mathfrak{B}_{<\mathcal{F}}$. In the group case, $|\Gamma_{\mathcal{F}}| = |\mathfrak{B}_{\mathcal{F}}|/|\mathfrak{B}_{<\mathcal{F}}|$.

For a level-0 fragment $\mathcal{F}^{[j,j+1]}$, we define the *nondynamical granule* $\Gamma_{\mathcal{F}}$ as $\mathfrak{B}^{[j,j+1]} \cong \underline{\mathcal{A}}_j$. The set $\{\Gamma_{\mathcal{F}}, \mathcal{F} \in \mathfrak{F}(\mathcal{R})\}$ thus consists of nondynamical granules at level $\ell = 0$, and controller granules at levels $\ell > 0$.

At level n , where $\mathcal{F} = \mathcal{R}$, we will call $\Gamma_{\mathcal{R}} = \mathfrak{B}/\mathfrak{B}_{<\mathcal{R}}$ the *top granule* of \mathcal{R} , or the *controllability granule* of \mathcal{R} , since as we will see $\Gamma_{\mathcal{R}}$ governs the controllability properties of \mathcal{R} .

Note that $\mathfrak{B}_{<\mathcal{R}} = \sum_j \mathfrak{B}^{[j,j]}$, the behavior generated by all level- $(n-1)$ subbehaviors $\mathfrak{B}^{[j,j]}$. We will call $\mathfrak{B}_{<\mathcal{R}}$ the *controllable subbehavior* \mathfrak{B}^c of \mathfrak{B} .

At levels $1 \leq \ell \leq n-1$, a proper fragment $\mathcal{F}^{[j,k]}$ covers precisely two fragments, namely $\mathcal{F}^{[j,k-1]}$ and $\mathcal{F}^{[j+1,k]}$. Thus $\mathfrak{B}_{<\mathcal{F}^{[j,k]}} = \mathfrak{B}^{[j,k-1]} + \mathfrak{B}^{[j+1,k]}$, and the corresponding controller granule is

$$\Gamma^{[j,k]} = \frac{\mathfrak{B}^{[j,k]}}{\mathfrak{B}^{[j,k-1]} + \mathfrak{B}^{[j+1,k]}}.$$

Forney and Trott [5], [6] define a controller granule for a conventional group trellis realization similarly as $\Gamma^{[j,k]} = \mathcal{C}^{[j,k]} / (\mathcal{C}^{[j,k-1]} + \mathcal{C}^{[j+1,k]})$, where the subcode $\mathcal{C}^{[j,k]} \subseteq \mathcal{C}$ is the set of $\mathbf{a} \in \mathcal{C}$ that are all-zero outside the boundary of $\mathcal{F}^{[j,k]}$. The two definitions turn out to be equivalent for minimal conventional trellis realizations.

D. ℓ -controllable behaviors

For $0 \leq \ell \leq n-1$, we define the ℓ -controllable behavior \mathfrak{B}_{ℓ} as the behavior generated by all level- ℓ subbehaviors $\mathfrak{B}^{[j,j+\ell]}$. In other words, $\mathfrak{B}_{\ell} = \sum_j \mathfrak{B}^{[j,j+\ell]}$. Note that $\mathfrak{B}_{n-1} = \mathfrak{B}^c$, the controllable subbehavior of \mathfrak{B} . We also define $\mathfrak{B}_n = \mathfrak{B}$.

Evidently $\mathfrak{B}_{\ell-1} \subseteq \mathfrak{B}_{\ell}$ for $1 \leq \ell \leq n$. Moreover, $\mathfrak{B}_0 = \sum_j \mathfrak{B}^{[j,j+1]} = \underline{\mathcal{A}} \times \{\mathbf{0}\}$, where $\underline{\mathcal{A}} = \{\mathbf{a} \in \mathcal{A} \mid (\mathbf{a}, \mathbf{0}) \in \mathfrak{B}\} = \prod_j \underline{\mathcal{A}}_j$. We call \mathfrak{B}_0 the *nondynamical behavior* of \mathcal{R} .

We thus have a chain of subgroups $\mathfrak{B}_0 = \underline{\mathcal{A}} \times \{\mathbf{0}\} \subseteq \mathfrak{B}_1 \subseteq \dots \subseteq \mathfrak{B}_n = \mathfrak{B}$, which is a normal series since all groups are abelian. We denote the factor groups of this chain by $Q_{\ell} = \mathfrak{B}_{\ell}/\mathfrak{B}_{\ell-1}$, $1 \leq \ell \leq n$, plus $Q_0 = \mathfrak{B}_0$.

By elementary group theory, we have $|\mathfrak{B}| = \prod_{\ell} |Q_{\ell}|$; or, in the linear case, $\dim \mathfrak{B} = \sum_{\ell} \dim Q_{\ell}$. If we define sets $[Q_{\ell}]$ of coset representatives for the cosets of $\mathfrak{B}_{\ell-1}$ in \mathfrak{B}_{ℓ} , then every $(\mathbf{a}, \mathbf{s}) \in \mathfrak{B}$ may be uniquely expressed as a sum of coset representatives; or, in the linear case, if we define a

basis \mathcal{B}_ℓ for each quotient Q_ℓ , then every $(\mathbf{a}, \mathbf{s}) \in \mathfrak{B}$ may be uniquely expressed as a linear combination of basis elements.

Since Q_ℓ is generated by the elements of \mathfrak{B}_ℓ that are not in $\mathfrak{B}_{\ell-1}$, and every element of \mathfrak{B}_ℓ is an element of some level- ℓ subbehavior $\mathfrak{B}^{[j,j+\ell]}$, the nonzero coset representatives in $[Q_\ell]$ may all be taken as elements of some $\mathfrak{B}^{[j,j+\ell]} \setminus \mathfrak{B}_{\ell-1}$. We note that if $(\mathbf{a}, \mathbf{s}) \in \mathfrak{B}^{[j,j+\ell]} \setminus \mathfrak{B}_{\ell-1}$, then the support of \mathbf{s} must be precisely the length- ℓ circular interval $[j+1, j+\ell]$, else $(\mathbf{a}, \mathbf{s}) \in \mathfrak{B}_{\ell-1}$.

The level- ℓ subbehaviors $\mathfrak{B}^{[j,j+\ell]}$ thus comprise a sufficient set of representatives for Q_ℓ . We say that *unique factorization* holds if every element of every level- ℓ behavior \mathfrak{B}_ℓ is a unique sum of elements of level- ℓ subbehaviors $\mathfrak{B}^{[j,j+\ell]}$, modulo $\mathfrak{B}_{\ell-1}$; i.e., if \mathfrak{B}_ℓ modulo $\mathfrak{B}_{\ell-1}$ is the (internal) *direct sum*

$$\mathfrak{B}_\ell = \bigoplus_{j \in \mathbb{Z}_n} \mathfrak{B}^{[j,j+\ell]} \quad \text{mod } \mathfrak{B}_{\ell-1}.$$

III. CONTROLLABILITY AND UNIQUE FACTORIZATION

In previous work [4], [3], we have defined controllability as the property of “having independent constraints,” since we have proved that a realization is observable if and only if its dual realization has this property.

We now show that for a linear or group tail-biting trellis realization \mathcal{R} , controllability in this sense is equivalent to the property that the top granule $\Gamma_{\mathcal{R}}$ is trivial. Simultaneously, we obtain an easy proof that unique factorization holds for \mathcal{R} , under the proviso (as in [8], [9], [2]) that \mathcal{R} is *reduced*; that is, \mathcal{R} is *state-trim*—i.e., $\mathfrak{B}_{|S_j} = S_j$ for all j —and \mathcal{R} is *branch-trim*—i.e., $\mathfrak{B}_{|S_j \times \mathcal{A}_j \times S_{j+1}} = C_j$ for all j .

(Notation: in this section, we will use notation appropriate to the group case—i.e., we use sizes rather than dimensions; the reader may translate to the linear case if desired.)

A. Controllability

In [4], [3], a realization \mathcal{R} is called *controllable* if the the constraints of \mathcal{U} and the equality constraints $\mathbf{s} = \mathbf{s}'$ are independent. More concretely, \mathcal{R} is controllable if the image S^c of the syndrome-former homomorphism $\mathcal{U} \rightarrow \mathcal{S}$ defined by $(\mathbf{s}, \mathbf{a}, \mathbf{s}') \mapsto \mathbf{s} - \mathbf{s}'$ is equal to \mathcal{S} . Since the kernel of this homomorphism is the extended behavior \mathfrak{B} , we have $\mathcal{U}/\mathfrak{B} \cong S^c \subseteq \mathcal{S}$, by the FTH. This yields the following **controllability test**: $|\mathcal{U}|/|\mathfrak{B}| \leq |\mathcal{S}|$, with equality if and only if \mathcal{R} is controllable [3]. In other words, since $\mathfrak{B} \cong \mathfrak{B}$, a realization is uncontrollable if and only if its constraints are dependent in the following sense:¹

$$|\mathfrak{B}| > \frac{|\mathcal{U}|}{|\mathcal{S}|} = \frac{\prod_j |C_j|}{\prod_j |S_j|}.$$

¹This result may be understood as follows. Ignoring state equality constraints, there are $|\mathcal{U}| = \prod_j |C_j|$ possible configurations. If the state equality constraints $\{s_j = s'_j, j \in \mathbb{Z}_n\}$ are all independent of the set of code constraints $\{C_j, j \in \mathbb{Z}_n\}$, then each state equality constraint $s_j = s'_j$ reduces the number of possible configurations by a factor of $|S_j|$, so $|\mathfrak{B}| = |\mathcal{U}|/|\mathcal{S}|$, where $|\mathcal{S}| = \prod_j |S_j|$. If the constraints are dependent—i.e., if \mathcal{R} is not controllable—then the reduction is strictly less, and $|\mathfrak{B}| > |\mathcal{U}|/|\mathcal{S}|$.

B. Disconnected trellis realizations

We now show that if the top granule $\Gamma_{\mathcal{R}} = \mathfrak{B}/\mathfrak{B}^c$ is nontrivial, then \mathfrak{B} consists of $|\Gamma_{\mathcal{R}}|$ disconnected subbehaviors, namely the cosets of the controllable subbehavior $\mathfrak{B}^c = \sum_j \mathfrak{B}^{[j,j]}$ in \mathfrak{B} . Similar results were proved in [4] and [7, Appendix A]; the proof here is simpler, and does not rely on duality.

Lemma. For a linear or group trellis realization \mathcal{R} with behavior \mathfrak{B} and controllable subbehavior \mathfrak{B}^c , for any $j \in \mathbb{Z}_n$:

- (a) $\mathfrak{B}_{|S_j}/(\mathfrak{B}^c)_{|S_j} \cong \Gamma_{\mathcal{R}}$;
- (b) $\mathfrak{B}_{|S_j \times \mathcal{A}_j \times S_{j+1}}/(\mathfrak{B}^c)_{|S_j \times \mathcal{A}_j \times S_{j+1}} \cong \Gamma_{\mathcal{R}}$.

Proof. (a) The projections of \mathfrak{B} and \mathfrak{B}^c onto S_j have a common kernel $\mathfrak{B}^{[j,j]} = \{(\mathbf{a}, \mathbf{s}) \in \mathfrak{B} \mid s_j = 0\}$. Thus $\mathfrak{B}_{|S_j}/(\mathfrak{B}^c)_{|S_j} \cong \mathfrak{B}/\mathfrak{B}^c = \Gamma_{\mathcal{R}}$, by the CT.

(b) The projections of \mathfrak{B} and \mathfrak{B}^c onto $S_j \times \mathcal{A}_j \times S_{j+1}$ have a common kernel $\mathfrak{B}^{[j+1,j]} = \{(\mathbf{a}, \mathbf{s}) \in \mathfrak{B} \mid (s_j, a_j, s_{j+1}) = (0, 0, 0)\}$, so (b) follows also from the CT. \square

If \mathcal{R} is reduced, as we assume, then $\mathfrak{B}_{|S_j} = S_j$ and $\mathfrak{B}_{|S_j \times \mathcal{A}_j \times S_{j+1}} = C_j$. Moreover, we may regard \mathfrak{B}^c as the behavior of the *controllable subrealization* \mathcal{R}^c of \mathcal{R} , defined as the reduced tail-biting trellis realization with state spaces $(S_j)^c = (\mathfrak{B}^c)_{|S_j}$, symbol spaces \mathcal{A}_j , and constraint codes $(C_j)^c = (\mathfrak{B}^c)_{|S_j \times \mathcal{A}_j \times S_{j+1}}$. This lemma then states that $S_j/(S_j)^c \cong \Gamma_{\mathcal{R}}$ and $C_j/(C_j)^c \cong \Gamma_{\mathcal{R}}$.

More concretely, (a) implies that, if $\Gamma_{\mathcal{R}}$ is nontrivial, then for each j , each coset $\mathfrak{B}^c + (\mathbf{a}, \mathbf{s})$ of \mathfrak{B}^c in \mathfrak{B} passes through a distinct corresponding coset $(S_j)^c + (\mathbf{s})_j$ of $(S_j)^c$ in S_j . Similarly, C_j partitions into $|\Gamma_{\mathcal{R}}|$ disjoint cosets of $(C_j)^c$, each representing state transitions within one coset of \mathfrak{B}^c in \mathfrak{B} . The trellis diagram of \mathcal{R} thus consists of $|\Gamma_{\mathcal{R}}|$ disconnected subdiagrams, one representing each coset of \mathfrak{B}^c in \mathfrak{B} . Thus for any j, j' , there is no trajectory (\mathbf{a}, \mathbf{s}) connecting any state s_j in a given coset of $(S_j)^c$ in S_j to a state $s_{j'}$ in a coset of $(S_{j'})^c$ in $S_{j'}$, unless the two cosets correspond to the same coset of \mathfrak{B}^c in \mathfrak{B} .

C. First-state chain

We now show that the controller granules of \mathcal{R} are isomorphic to factor groups of certain normal series.

Lemma (first-state chain). For $j \in \mathbb{Z}_n$, $1 \leq \ell \leq n-1$,

$$\Gamma^{[j,j+\ell]} \cong \frac{(\mathfrak{B}^{[j,j+\ell]})_{|S_j \times \mathcal{A}_j \times S_{j+1}}}{(\mathfrak{B}^{[j,j+\ell]})_{|S_j \times \mathcal{A}_j \times S_{j+1}}} \cong \frac{(\mathfrak{B}^{[j,j+\ell]})_{|S_{j+1}}}{(\mathfrak{B}^{[j,j+\ell]})_{|S_{j+1}}}.$$

Proof. We have $\Gamma^{[j,j+\ell]} = \mathfrak{B}^{[j,j+\ell]}/(\mathfrak{B}^{[j,j+\ell]} + \mathfrak{B}^{(j,j+\ell)})$. The projections of $\mathfrak{B}^{[j,j+\ell]}$ and $\mathfrak{B}^{[j,j+\ell]} + \mathfrak{B}^{(j,j+\ell)}$ onto $S_j \times \mathcal{A}_j \times S_{j+1}$ are $(\mathfrak{B}^{[j,j+\ell]})_{|S_j \times \mathcal{A}_j \times S_{j+1}}$ and $(\mathfrak{B}^{[j,j+\ell]})_{|S_j \times \mathcal{A}_j \times S_{j+1}}$, respectively, and their common kernel is $\mathfrak{B}^{(j,j+\ell)} = \{(\mathbf{a}, \mathbf{s}) \in \mathfrak{B}^{[j,j+\ell]} \mid (s_j, a_j, s_{j+1}) = (0, 0, 0)\}$. Similarly, the projections of $(\mathfrak{B}^{[j,j+\ell]})_{|S_j \times \mathcal{A}_j \times S_{j+1}}$ and $(\mathfrak{B}^{[j,j+\ell]})_{|S_j \times \mathcal{A}_j \times S_{j+1}}$ onto S_{j+1} are $(\mathfrak{B}^{[j,j+\ell]})_{|S_{j+1}}$ and $(\mathfrak{B}^{[j,j+\ell]})_{|S_{j+1}}$, respectively, and their common kernel is $(\mathfrak{B}^{[j,j]})_{|S_j \times \mathcal{A}_j \times S_{j+1}} = \{0\} \times \mathcal{A}_j \times \{0\}$. Thus both isomorphisms follow from the CT. \square

It follows from the first isomorphism that for each \mathcal{C}_j there is a normal series $(\mathfrak{B}^{[j,j]})_{|\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}|} = \{0\} \times \underline{\mathcal{A}}_j \times \{0\} \subseteq (\mathfrak{B}^{[j,j+1]})_{|\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}|} \subseteq \dots \subseteq (\mathfrak{B}^{[j,j]})_{|\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}|}$, whose factor groups are isomorphic to the granules $\Gamma^{[j,j+\ell]}$, $0 \leq \ell \leq n-1$. This chain implies that

$$|(\mathfrak{B}^{[j,j]})_{|\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}|}| = \prod_{\ell=0}^{n-1} |\Gamma^{[j,j+\ell]}|.$$

This result will be useful in the next section.

It follows from the second isomorphism that for each state space \mathcal{S}_{j+1} there is a normal series $(\mathfrak{B}^{[j,j]})_{|\mathcal{S}_{j+1}|} = \{0\} \subseteq (\mathfrak{B}^{[j,j+1]})_{|\mathcal{S}_{j+1}|} \subseteq \dots \subseteq (\mathfrak{B}^{[j,j]})_{|\mathcal{S}_{j+1}|}$, whose factor groups are isomorphic to the granules $\Gamma^{[j,j+\ell]}$, $1 \leq \ell \leq n-1$. We call this normal series the *first-state chain* at \mathcal{S}_{j+1} , since \mathcal{S}_{j+1} is the first possibly nonzero state in the trajectories in $\mathfrak{B}^{[j,j+\ell]}$, $1 \leq \ell \leq n-1$. This chain implies that

$$|(\mathfrak{B}^{[j,j]})_{|\mathcal{S}_{j+1}|}| = \prod_{\ell=1}^{n-1} |\Gamma^{[j,j+\ell]}|.$$

D. Controllability and unique factorization

We will now show that \mathcal{R} is controllable if and only if $\mathfrak{B} = \mathfrak{B}^c$; i.e., if and only if the top granule $\Gamma_{\mathcal{R}}$ is trivial. Moreover, the controller granule decomposition gives a unique factorization of both \mathfrak{B}^c and \mathfrak{B} .

We first state a technical lemma that shows that in the controllable subrealization \mathcal{R}^c , the number of transitions $(s_j, a_j, s_{j+1}) \in (\mathcal{C}_j)^c$ is the number of states $s_j \in (\mathcal{S}_j)^c$ times the number of transitions $(0, a_j, s_{j+1}) \in (\mathfrak{B}^{[j,j]})_{|\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}|}$.

Lemma. For all j , $|(\mathcal{C}_j)^c| = |(\mathcal{S}_j)^c| \cdot |(\mathfrak{B}^{[j,j]})_{|\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}|}|$.

Proof. The projection of \mathfrak{B}^c on \mathcal{S}_j is $(\mathcal{S}_j)^c$, and its kernel is $\mathfrak{B}^{[j,j]}$, so $(\mathcal{S}_j)^c \cong \mathfrak{B}^c / \mathfrak{B}^{[j,j]}$ by the FTH. The projections of \mathfrak{B}^c and $\mathfrak{B}^{[j,j]}$ on $\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}$ are $(\mathcal{C}_j)^c$ and $(\mathfrak{B}^{[j,j]})_{|\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}|}$, respectively, and $\mathfrak{B}^{[j,j+1]}$ is their common kernel, so $\mathfrak{B}^c / \mathfrak{B}^{[j,j]} \cong (\mathcal{C}_j)^c / (\mathfrak{B}^{[j,j]})_{|\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}|}$ by the CT. \square

Next, we define P^c as the product of the sizes of all controller granules up to level $n-1$, i.e., $P^c = \prod_{\ell=0}^{n-1} \prod_{j \in \mathbb{Z}_n} |\Gamma^{[j,j+\ell]}|$, and $P = |\Gamma_{\mathcal{R}}| P^c$ as the product of the sizes of all controller granules. We observe that since P is the number of possible sums of granule representatives, we have $|\mathfrak{B}| \leq P$, with equality if and only if unique factorization holds for \mathfrak{B} . Similarly, we have $|\mathfrak{B}^c| \leq P^c$, with equality if and only if unique factorization holds for \mathfrak{B}^c .

Theorem (controllability and unique factorization). Let \mathfrak{B} and \mathfrak{B}^c be the behaviors of a reduced linear or group tail-biting trellis realization \mathcal{R} and its controllable subrealization \mathcal{R}^c , respectively. Then:

- (a) \mathcal{R}^c is controllable.
- (b) Unique factorization holds for \mathfrak{B}^c ; i.e., $|\mathfrak{B}^c| = P^c$.
- (c) \mathcal{R} is controllable if and only if $\mathfrak{B} = \mathfrak{B}^c$; i.e., iff the top granule $\Gamma_{\mathcal{R}}$ is trivial.
- (d) Unique factorization holds for \mathfrak{B} ; i.e., $|\mathfrak{B}| = P$.

Proof. (a-b) From the previous lemma, $\prod_j |(\mathcal{C}_j)^c| = (\prod_j |(\mathcal{S}_j)^c|) (\prod_j |(\mathfrak{B}^{[j,j]})_{|\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}|}|)$. By Section III-C, we have $|(\mathfrak{B}^{[j,j]})_{|\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}|}| = \prod_{\ell=0}^{n-1} |\Gamma^{[j,j+\ell]}|$, so $(\prod_j |(\mathcal{C}_j)^c|) / (\prod_j |(\mathcal{S}_j)^c|) = \prod_j \prod_{\ell=0}^{n-1} |\Gamma^{[j,j+\ell]}| = P^c$, the product of the sizes of all proper controller granules $\Gamma^{[j,j+\ell]}$. Therefore, by our controllability test, we have $|\mathfrak{B}^c| \geq P^c$, with equality if and only if \mathcal{R}^c is controllable. On the other hand, in view of the controller granule decomposition of \mathfrak{B}^c , we have $|\mathfrak{B}^c| \leq P^c$, with equality if and only if unique factorization holds for \mathfrak{B}^c . Thus $|\mathfrak{B}^c| = P^c$, \mathcal{R}^c is controllable, and unique factorization holds for \mathfrak{B}^c .

(c) By Section III-B, \mathfrak{B} is the disjoint union of $|\Gamma_{\mathcal{R}}|$ disconnected cosets of \mathfrak{B}^c . Thus we have $|\mathfrak{B}| = |\Gamma_{\mathcal{R}}| |\mathfrak{B}^c|$, $|\mathcal{C}_j| = |\Gamma_{\mathcal{R}}| |(\mathcal{C}_j)^c|$, and $|\mathcal{S}_j| = |\Gamma_{\mathcal{R}}| |(\mathcal{S}_j)^c|$. Therefore $(\prod_j |\mathcal{C}_j|) / (\prod_j |\mathcal{S}_j|) = P^c = |\mathfrak{B}^c| = |\mathfrak{B}| / |\Gamma_{\mathcal{R}}|$. By our controllability test, \mathcal{R} is controllable if and only if $|\Gamma_{\mathcal{R}}| = 1$.

(d) By Section III-B, every element of \mathfrak{B} is uniquely expressible as the sum of an element of \mathfrak{B}^c and a coset representative in $[\Gamma_{\mathcal{R}}]$, so since unique factorization holds for \mathfrak{B}^c , it holds also for \mathfrak{B} . \square

E. State space and constraint code sizes

Unique factorization of \mathfrak{B} implies unique factorization of $\mathfrak{B}_{\mathcal{F}}$ for any fragment $\mathcal{F} \leq \mathcal{R}$. It follows that the size of each state space \mathcal{S}_j and each constraint code \mathcal{C}_j may be determined in terms of granule sizes as follows:

Corollary (state space and constraint code sizes). If \mathcal{R} is a reduced linear or group tail-biting trellis realization with state spaces \mathcal{S}_j and constraint codes \mathcal{C}_j , then:

- (a) $\mathcal{S}_j \cong \mathfrak{B} / \mathfrak{B}^{[j,j]}$, and

$$|\mathcal{S}_j| = \prod_{\mathcal{F} \leq \mathcal{R}: \mathcal{S}_j \in E(\mathcal{F})} |\Gamma_{\mathcal{F}}|;$$

- (b) $\mathcal{C}_j \cong \mathfrak{B} / \mathfrak{B}^{[j+1,j]}$, and

$$|\mathcal{C}_j| = \prod_{\mathcal{F} \leq \mathcal{R}: \mathcal{C}_j \in V(\mathcal{F})} |\Gamma_{\mathcal{F}}|.$$

Proof. (a) If \mathcal{R} is state-trim at \mathcal{S}_j , then $\mathcal{S}_j = \mathfrak{B}_{|\mathcal{S}_j|}$. Moreover, the kernel of the projection of \mathfrak{B} onto \mathcal{S}_j is $\mathfrak{B}^{[j,j]}$. Thus $\mathcal{S}_j \cong \mathfrak{B} / \mathfrak{B}^{[j,j]}$ by the FTH, so $|\mathcal{S}_j| = |\mathfrak{B}| / |\mathfrak{B}^{[j,j]}| = P / \prod_{\mathcal{F} \leq \mathcal{R}: \mathcal{S}_j \in E(\mathcal{F})} |\Gamma_{\mathcal{F}}| = \prod_{\mathcal{F} \leq \mathcal{R}: \mathcal{S}_j \in E(\mathcal{F})} |\Gamma_{\mathcal{F}}|$, since $\mathcal{F} \leq \mathcal{R}$ iff $\mathcal{S}_j \in E(\mathcal{F})$.

(b) If \mathcal{R} is branch-trim at \mathcal{C}_j , then $\mathcal{C}_j = \mathfrak{B}_{|\mathcal{S}_j \times \mathcal{A}_j \times \mathcal{S}_{j+1}|}$. Moreover, the kernel of the projection of \mathfrak{B} onto \mathcal{C}_j is $\mathfrak{B}^{[j+1,j]}$. Thus $\mathcal{C}_j \cong \mathfrak{B} / \mathfrak{B}^{[j+1,j]}$ by the FTH, so $|\mathcal{C}_j| = |\mathfrak{B}| / |\mathfrak{B}^{[j+1,j]}| = P / \prod_{\mathcal{F} \leq \mathcal{R}: \mathcal{C}_j \in V(\mathcal{F})} |\Gamma_{\mathcal{F}}| = \prod_{\mathcal{F} \leq \mathcal{R}: \mathcal{C}_j \in V(\mathcal{F})} |\Gamma_{\mathcal{F}}|$, since $\mathcal{F} \leq \mathcal{R}$ iff $\mathcal{C}_j \in V(\mathcal{F})$. \square

In other words, assuming trimness, \mathcal{S}_j factors into components isomorphic to those granules $\Gamma_{\mathcal{F}}$ such that $\mathcal{S}_j \in E(\mathcal{F})$ (i.e., \mathcal{S}_j is “active” during \mathcal{F}). Also, \mathcal{C}_j factors into components isomorphic to those granules $\Gamma_{\mathcal{F}}$ such that $\mathcal{C}_j \in V(\mathcal{F})$ (i.e., \mathcal{C}_j is “active” during \mathcal{F}).

F. Controller canonical realization

The unique factorization result of Section III-D implies that every reduced linear or group trellis realization is equivalent to a *controller canonical realization*, which we define as follows.

For each $\mathcal{F} \leq \mathcal{R}$, we have a one-to-one map $\Gamma_{\mathcal{F}} \rightarrow [\Gamma_{\mathcal{F}}]$ from the granule $\Gamma_{\mathcal{F}}$ to the set of coset representatives $[\Gamma_{\mathcal{F}}] = [\mathfrak{B}_{\mathcal{F}}/\mathfrak{B}_{<\mathcal{F}}]$. We may thus map each element of the Cartesian product $\prod_{\mathcal{F} \leq \mathcal{R}} \Gamma_{\mathcal{F}}$ to the sum $(\mathbf{a}, \mathbf{s}) = \sum_{\mathcal{F} \leq \mathcal{R}} (\mathbf{a}_{\mathcal{F}}, \mathbf{s}_{\mathcal{F}})$ of the corresponding coset representatives $(\mathbf{a}_{\mathcal{F}}, \mathbf{s}_{\mathcal{F}}) \in [\Gamma_{\mathcal{F}}]$, which is an element of \mathfrak{B} since each coset representative is an element of \mathfrak{B} . By unique factorization, the map so defined from $\prod_{\mathcal{F} \leq \mathcal{R}} \Gamma_{\mathcal{F}}$ to \mathfrak{B} is one-to-one.

More concretely, the map $\prod_{\mathcal{F} \leq \mathcal{R}} \Gamma_{\mathcal{F}} \rightarrow \mathfrak{B}$ may be implemented as follows. We generate the trajectories in $[\Gamma_{\mathcal{F}}]$ by an *atomic trellis realization* whose state spaces \mathcal{S}_j are equal to $\Gamma_{\mathcal{F}}$ when $\mathcal{S}_j \in E(\mathcal{F})$, and trivial otherwise. An element of $\Gamma_{\mathcal{F}}$ determines the state value $(\mathbf{s}_{\mathcal{F}})_j$ when $\mathcal{S}_j \in E(\mathcal{F})$, and the symbol value $(\mathbf{a}_{\mathcal{F}})_j$ when $\mathcal{C}_j \in V(\mathcal{F})$. The state value \mathbf{s}_j is thus the sum $\sum_{\mathcal{F} \leq \mathcal{R} | \mathcal{S}_j \in E(\mathcal{F})} (\mathbf{s}_{\mathcal{F}})_j$, and the symbol value \mathbf{a}_j is the sum $\sum_{\mathcal{F} \leq \mathcal{R} | \mathcal{C}_j \in V(\mathcal{F})} (\mathbf{a}_{\mathcal{F}})_j$. The size of the aggregate state space \mathcal{S}_j is thus $|\mathcal{S}_j| = \prod_{\mathcal{F} \leq \mathcal{R} | \mathcal{S}_j \in E(\mathcal{F})} |\Gamma_{\mathcal{F}}|$, as in our state space size result. Thus the controller canonical realization is a minimal realization of \mathfrak{B} . (We can also show that the number of possible transitions (s_j, a_j, s_{j+1}) is $\prod_{\mathcal{F} \leq \mathcal{R} | \mathcal{C}_j \in V(\mathcal{F})} |\Gamma_{\mathcal{F}}|$, as in our constraint code size result.)

If \mathfrak{B} is linear, then the controller canonical realization of \mathfrak{B} is easily seen to be linear. However, for a group realization \mathcal{R} , although the map $\prod_{\mathcal{F} \leq \mathcal{R}} \Gamma_{\mathcal{F}} \rightarrow \mathfrak{B}$ yields a one-to-one, group-theoretic, and minimal realization of \mathfrak{B} , it may well not be isomorphic, even when \mathcal{R} is conventional [5]. This issue was raised in [2, Remark III.3] via the following example, in which the controller canonical realization is nonhomomorphic.

Example (Conventional group trellis realization over \mathbb{Z}_4). Let \mathcal{R} be a conventional group trellis realization of length 3 with behavior $\mathfrak{B} = \langle (112, 0120) \rangle \subseteq (\mathbb{Z}_4)^3 \times (\mathbb{Z}_4)^4$; i.e., $\mathfrak{B} = \{(000, 0000), (112, 0120), (220, 0200), (332, 0320)\} \cong \mathbb{Z}_4$. Its ℓ -controllable subbehaviors are $\mathfrak{B}_0 = \{(000, 0000)\}$; $\mathfrak{B}_1 = \mathfrak{B}^{[0,1]} = \{(000, 0000), (220, 0200)\} \cong 2\mathbb{Z}_4 \cong \mathbb{Z}_2$; and $\mathfrak{B}_2 = \mathfrak{B} \cong \mathbb{Z}_4$. Its nontrivial controller granules are $\Gamma^{[0,1]} = \mathfrak{B}^{[0,1]} \cong \mathbb{Z}_2$, which is realized by a 2-state atomic trellis realization that is active during $[0, 1]$, and $\Gamma^{[0,2]} = \mathfrak{B}/\mathfrak{B}^{[0,1]} \cong \mathbb{Z}_4/2\mathbb{Z}_4 \cong \mathbb{Z}_2$, which is realized by a 2-state atomic trellis realization that is active during $[0, 2]$.

Figure 2 depicts the controller canonical realization of \mathfrak{B} via trellis diagrams for the atomic trellis realizations of $\Gamma^{[0,1]} = \mathfrak{B}^{[0,1]}$ and $[\Gamma^{[0,2]}] = [\mathfrak{B}/\mathfrak{B}^{[0,1]}]$, plus a trellis diagram for \mathfrak{B} .

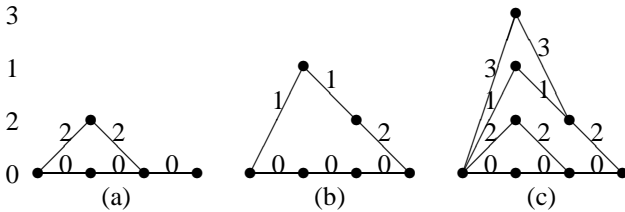


Fig. 2. Trellis diagrams for (a) $\Gamma^{[0,1]}$; (b) $\Gamma^{[0,2]}$; (c) \mathfrak{B} . \square

IV. CONCLUSION

We have generalized the CBFT to group trellis realizations, with a proof based on a controller granule decomposition of \mathfrak{B} and our controllability test for general group realizations.

It would be natural to dualize these results, using a dual observer granule decomposition. However, as discussed in [6], such a dualization is not straightforward, even for minimal conventional trellis realizations. Developing a nice dual observer granule decomposition for linear and group tail-biting trellis realizations is a good goal for future research.

It would be nice also to extend these results to non-trellis realizations. However, it is known (see [4, Appendix A]) that unique factorization generally does not hold for non-trellis linear or group realizations, even simple cycle-free realizations. New ideas will therefore be needed.

Finally, we would like ultimately to redevelop all of the principal results of classical discrete-time linear systems theory using a purely group-theoretic approach. However, the classical theory generally assumes an infinite time axis. One possible approach would be to regard a time-invariant or periodically time-varying linear or group system on an infinite time axis as the “limit” of a sequence of covers of a linear or group tail-biting trellis realization on a sequence of finite time axes of increasing length. Such an approach would hopefully be purely algebraic, and thus might avoid the subtle topological issues discussed in [6].

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